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A NOTE ON APPELL-TYPE λ-DAEHEE-HERMITE POLYNOMIALS AND NUMBERS

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Abstract. In this paper, we introduce a new class of λ-analogue of the Daehee-Hermite polynomials and generalized Gould-Hopper-Appell type λ-Daehee polynomials and present some properties and identities of these polynomials. A new class of polynomials generalizing different classes of Hermite polynomials such as the real Gould-Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex Hermite polynomials and their relationship to the Appell type λ-Daehee polynomials are also discussed.

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1 Introduction

Let p be a fixed odd prime number. Throughout the article, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $\mid p \mid_p = p^{\nu_p(p)} = \frac{1}{p}$. For $\cup D(\mathbb{Z}_p)$ be space of uniformly differentiable function on $\mathbb{Z}_P$. For $f \in \cup D(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_0(\xi) = \lim_{N\to\infty} \sum_{\xi=0}^{p^N-1} f(\xi)\mu_0(\xi + p^N\mathbb{Z}_p) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{\xi=0}^{p^N-1} f(\xi). \tag{1}$$

It is apparent from (1) that $I_0(f_1) = I_0(f)+f'(0)$, (Khan et al., 2022; Kim, 2013; Kim & Kim, 2013), where $f_1(\xi) = f(\xi+1)$.

The Daehee polynomials are defined by the generating function

$$\frac{\log(1+z)}{z}(1+z)^\xi = \sum_{j=0}^{\infty} D_j(\xi)\frac{z^j}{j!}, \tag{2}$$

(Khan et al., 2022).

In the case $\xi = 0$, $D_j = D_j(0)$ are the Daehee numbers.

The two variable Hermite Kampé de Fériet polynomials $H_j(\xi,\eta)$ (Andrews, 1985; Bell, 1934) are defined by

$$H_j(\xi,\eta) = j!\sum_{r=0}^{[\frac{j}{2}]} \frac{\eta^r \xi^{j-2r}}{r!(j-2r)!}. \tag{3}$$

These polynomials are usually defined by the generating function

$$e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} H_j(\xi,\eta)\frac{z^j}{j!}, \tag{4}$$

and reduce to the ordinary Hermite polynomials $H_j(\xi)$ (see (Andrews, 1985)) when $\eta = -1$ and ξ is replaced by 2ξ.

Jedda & Ghanmi (2014) introduced a class of two-index real Hermite polynomials of degree $p+j$ by

$$h_{p,j}(\xi) = (-\frac{d}{d\xi}+2\xi)^p(\xi)^j = p!j!\sum_{k=0}^{min(p,j)} \frac{(-1)^k \xi^{p-k} H_{p-k}(\xi)}{k!(j-k)!(p-k)}. \tag{5}$$

Note that $h_{p,0}(\xi) = H_p\xi(\xi),\ h_{0,j\xi}(\xi) = \xi^j$ and $h_{p,j}(0) = 0$, if $p < j$.

The generating function of $h_{p,j}$ is given by

$$\sum_{p=0}^{\infty}\sum_{j=0}^{\infty} h_{p,j}(\xi)\frac{u^p v^j}{p!j!} = e^{-u^2+(2u+v)\xi-uv}. \tag{6}$$

Furthermore, for $\eta = u = -v$, (Jedda & Ghanmi, 2014) proved that

$$e^{\xi\eta} = \sum_{p,j=0}^{\infty} (-1)^j h_{p,j}(\xi)\frac{\eta^{p+j}}{p!j!}. \tag{7}$$

The generating function of Gould-Hopper polynomials $G_m^{(q)}(w|\gamma)$ introduced by Dattoli et al. (1994), p. 72 is given by

$$e^{wv+\gamma v^q} = \sum_{m=0}^{\infty} G_m^{(q)}(w|\gamma)\frac{v^m}{m!}, \tag{8}$$

so that for every complex numbers u, v, z and w, we have (see Ghanmi & Lamsaf (2019), pages (5) and (6)):

$$\sum_{j=0}^{\infty} H_{j,m}^{(p,q)}(z,w|\gamma)\frac{u^n}{n!} = G_m^{(q)}(w|u^p\gamma)e^{zu}, \tag{9}$$

and

$$\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(z,w|\gamma)\frac{v^m}{m!} = G_j^{(q)}(w|v^p\gamma)e^{zv}, \tag{10}$$

where the polynomials $H_{j,m}^{(p,q)}(z,w|\gamma)$ contain all the classes given above. Moreover, they give rise to new classes of polynomials of Hermite type. The concrete study of this polynomial is presented in (Ghanmi & Lamsaf, 2019) in a unified way and includes the connection to Gould-Hopper polynomials (Gould & Hopper, 1962), operational representations and connection to hypergeometric function, generating functions, addition formulas of Runge type, multiplication formulas, Nielson formulas and higher order differential equation they obey.

As is well known, the Bernoulli polynomials are defined by the generating function

$$\frac{z}{e^z-1}e^{\xi z} = \sum_{j=0}^{\infty} B_j(\xi)\frac{z^j}{j!}, \tag{11}$$

(see Andrews (1985); Khan et al. (2021a,b, 2020); Khan, W.A. (2022); Khan et al. (2022)).

When $\xi = 0$, $B_j = B_j(0)$ are called the Bernoulli numbers.

The Bernoulli polynomials of the second kind are defined by the generating function

$$\frac{z}{\log(1+z)}(1+z)^{\xi} = \sum_{j=0}^{\infty} b_j(\xi)\frac{z^j}{j!}, \tag{12}$$

(see Khan et al. (2021a,b)).

At the point $\xi = 0$, $b_j = b_j(0)$ are called the Bernoulli numbers of the second kind.

For $\lambda, z \in \mathbb{C}_p$ with $\mid \lambda z \mid_p < p^{-\frac{1}{p-1}}$, the Appell-type Daehee polynomials are defined by the generating function (see Kwon et al. (2015))

$$\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}e^{\xi z} = \sum_{j=0}^{\infty} D_j(\xi|\lambda)\frac{z^j}{j!}. \tag{13}$$

When $\xi = 0$, $D_j(\lambda)$=$D_j(0/\lambda)$ are called the Appell-type Daehee numbers.

Kwon et al. (2015) proved that

$$D_j^{(r)}(\xi|\lambda) = \sum_{s=0}^{j} \left(\begin{array}{c} j \\ s \end{array} \right) D_s^{(r)}(\lambda)\xi^{j-s}.$$

where

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z} = \sum_{j=0}^{\infty} D_j^{(r)}(\xi|\lambda)\frac{z^j}{j!}. \tag{14}$$

The falling factorial sequence is defined by

$$(\xi)_0 = 1, \quad (\xi)_j = \xi(\xi-1)\cdots(\xi-j+1), (j \geq 1). \tag{15}$$

The first kind of Stirling numbers are defined by

$$(\xi)_j = \sum_{k=0}^{j} S_1(j,k)\xi^k, (j \geq 0), \tag{16}$$

(see Andrews (1985); Bell (1934); Dattoli et al. (1994, 2003); Deeba & Rodrigues (1991); Gori (1994); Ghanmi & Lamsaf (2019); Gould & Hopper (1962); Jedda & Ghanmi (2014); Khan et al. (2021a)) and as an inversion formula of (15), the Stirling numbers of the second kind are given by (see Dattoli et al. (2003); Deeba & Rodrigues (1991); Gori (1994); Ghanmi & Lamsaf (2019); Gould & Hopper (1962); Jedda & Ghanmi (2014); Khan et al. (2021a,b, 2020); Khan, W.A. (2022)):

$$\xi^j = \sum_{k=0}^{j} S_2(j,k)(\xi)_k. \tag{17}$$

From (15) and (16), we note that the generating function of Stirling numbers of the first kind and that of the second kind are respectively given by (see Andrews (1985); Bell (1934); Dattoli et al. (1994, 2003); Deeba & Rodrigues (1991); Gori (1994); Ghanmi & Lamsaf (2019); Gould & Hopper (1962); Jedda & Ghanmi (2014); Khan et al. (2021a,b, 2022); Kim (2013); Kim & Kim (2013); Kim et al. (2014); Kwon et al. (2015); Pathan et al. (2015); Pathan & Khan (2016, 2021); 2022 (2022a,b)):

$$\frac{1}{k!}(\log(1+z))^k = \sum_{j=k}^{\infty} S_1(j,k)\frac{z^j}{j!}, \tag{18}$$

and

$$\frac{1}{k!}(e^z-1)^k=\sum_{j=k}^{\infty}S_2(j,k)\frac{z^j}{j!},(k\geq 0). \tag{19}$$

For each $p\geq 0$, $S_p(j)$ (Deeba & Rodrigues, 1991) defined by

$$S_p(j)=\sum_{l=0}^{j}l^p, \tag{20}$$

is called the sum of integer power sum or simply powers sum. The exponential generating function for $S_p(j)$ is

$$\sum_{p=0}^{\infty}S_p(j)\frac{z^p}{p!}=1+e^z+e^{2z}+\cdots+e^{jz}=\frac{e^{(j+1)z}-1}{e^z-1}. \tag{21}$$

In this paper, we have presented the generalized Appell type λ-Daehee-Hermite polynomials and discussed, in particular, some interesting series representations. We have deduced some relevant properties by using the structure and the relations satisfied by the recently generalized Hermite polynomials. Section 2 incorporates the definition of Appell type λ-Daehee-Hermite polynomials and a preliminary study of these polynomials. Some theorems on implicit summation formulae for Appell type λ-Daehee-Hermite polynomials ${}_HD_j^{(r)}(\xi,\eta|\lambda)$ and their special cases are given in Section 3. Section 4 is a consequence of the definition of the two-index real Hermite-Appell type λ-Daehee polynomials and generalized Gould-Hopper-Appell type λ-Daehee polynomials combined with their properties and special cases. Finally, symmetry identities for Appell type λ-Daehee-Hermite polynomials are given in Section 5.

2 Appell-type λ-Daehee-Hermite polynomials

In this section, we introduce the definition of Appell type λ-Daehee-Hermite polynomials and investigate some properties of these polynomials. First, we present the following definitions as.

Let as assume that $\lambda,z\in\mathbb{C}_p$ with $\mid\lambda z\mid_p< p^{-\frac{1}{p-1}}$. We define Appell-type λ-Daehee-Hermite polynomials as

$$\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}e^{\xi z+\eta z^2}=\sum_{j=0}^{\infty}{}_HD_j(\xi,\eta|\lambda)\frac{z^j}{j!}. \tag{22}$$

When $\xi=\eta=0$, $D_j(\lambda)={}_HD_j(0,0|\lambda)$ are called the Appell-type λ-Daehee numbers. For $\eta=0$ then ${}_HD_j(\xi,0|\lambda)=D_j(\xi|\lambda)$ and Appell-type λ-Daehee-Hermite polynomials reduce to Appell-type λ-Daehee polynomials.

Using (4), we can write (22) in the form

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z+\eta z^2}=\sum_{j=0}^{\infty}{}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}. \tag{23}$$

Theorem 1. *For $j\geq 0$. Then*

$${}_HD_j(\xi,\eta|\lambda)=\sum_{l=0}^{j}\left(\begin{array}{c} j \\ l \end{array}\right)D_l(\lambda)H_{j-l}(\xi,\eta). \tag{24}$$

Proof. By using (4) and (22), we complete the proof. □

Theorem 2. *For $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\xi,\eta|\lambda) = \sum_{l=0}^{[\frac{j}{2}]} \left(\begin{array}{c} j \\ 2l \end{array} \right) D_{j-2l}^{(r)}(\xi|\lambda)\eta^l. \tag{25} $$

Proof. In (23), we expand $e^{\eta z^2}$ in series, use (4) and then compare the coefficients of z on both the sides to get the result. □

Theorem 3. *For $j \geq 0$. Then*

$$ H_j(\xi,\eta) = \sum_{s=0}^{j}\sum_{l=0}^{s} \left(\begin{array}{c} s \\ l \end{array} \right) \left(\begin{array}{c} j \\ s \end{array} \right) \lambda^l D_l b_{s-l\,H}D_{j-s}(\xi,\eta|\lambda). \tag{26} $$

Proof. By the definition (22), we have

$$ \frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} {}_HD_j(\xi,\eta|\lambda)\frac{z^j}{j!}. \tag{27} $$

Since

$$ e^{\xi z+\eta z^2} = \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{\log(1+z)} \sum_{j=0}^{\infty} {}_HD_j(\xi,\eta|\lambda)\frac{z^j}{j!} $$

$$ = \frac{\log(1+\lambda z)}{\lambda z}\frac{z}{\log(1+z)} \sum_{j=0}^{\infty} {}_HD_j(\xi,\eta|\lambda)\frac{z^j}{j!} $$

$$ = \left(\sum_{l=0}^{\infty} D_l\frac{\lambda^l z^l}{l!}\right)\left(\sum_{s=0}^{\infty} b_s\frac{z^s}{s!}\right)\left(\sum_{j=0}^{\infty} {}_HD_j(\xi,\eta|\lambda)\frac{z^j}{j!}\right) $$

$$ = \left(\sum_{j=0}^{\infty}\left(\sum_{l=0}^{s} \left(\begin{array}{c} s \\ l \end{array} \right) \lambda^l D_l b_{s-l}\right)\frac{z^s}{s!}\right)\left(\sum_{j=0}^{\infty} {}_HD_j(\xi,\eta|\lambda)\frac{z^j}{j!}\right) $$

$$ = \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\sum_{l=0}^{s} \left(\begin{array}{c} s \\ l \end{array} \right) \left(\begin{array}{c} j \\ s \end{array} \right) \lambda^l D_l b_{s-l\,H}D_{j-s}(\xi,\eta|\lambda)\right)\frac{z^j}{j!} $$

$$ \sum_{j=0}^{\infty} H_j(\xi,\eta)\frac{z^j}{j!} $$

$$ = \sum_{j=0}^{\infty}\left(\sum_{s=0}^{j}\sum_{l=0}^{s} \left(\begin{array}{c} s \\ l \end{array} \right) \left(\begin{array}{c} j \\ s \end{array} \right) \lambda^l D_l b_{s-l\,H}D_{j-s}(\xi,\eta|\lambda)\right)\frac{z^j}{j!}. \tag{28} $$

Now comparing the coefficients of $\frac{z^j}{j!}$ in above equation, we get the required result. □

Theorem 4. *For $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\xi,\eta|\lambda) = \sum_{k=0}^{j}\sum_{s=0}^{k} \left(\begin{array}{c} k \\ s \end{array} \right) \left(\begin{array}{c} j \\ k \end{array} \right) D_s^{(r)}\lambda^{k-s}B_{k-s}^{(k-r+1)}(1)H_{j-s}(\xi,\eta).(29) $$

Proof. By using the definition (4) and (23), we have

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}.$$

Now

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z+\eta z^2} = \left(\frac{\log(1+z)}{z}\right)^r \left(\frac{\lambda z}{\log(1+\lambda z)}\right)^r e^{\xi z+\eta z^2}$$

$$= \left(\sum_{s=0}^{\infty} D_s^{(r)}\frac{z^s}{s!}\right)\left(\sum_{k=0}^{\infty} B_k^{(k-r+1)}(1)\frac{\lambda^k z^k}{k!}\right)\left(\sum_{j=0}^{\infty} H_j(\xi,\eta)\frac{z^j}{j!}\right)$$

$$= \left(\sum_{k=0}^{\infty}\sum_{s=0}^{k}\left(\begin{array}{c} k \\ s \end{array}\right) D_s^{(r)}\lambda^{k-s}B_{k-s}^{(k-r+1)}(1)\frac{z^k}{k!}\right)\left(\sum_{j=0}^{\infty} H_j(\xi,\eta)\frac{z^j}{j!}\right)$$

$$=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{s=0}^{k}\left(\begin{array}{c} k \\ s \end{array}\right)\left(\begin{array}{c} j \\ k \end{array}\right) D_s^{(r)}\lambda^{k-s}B_{k-s}^{(k-r+1)}(1)H_{j-s}(\xi,\eta)\right)\frac{z^j}{j!}. \quad (30)$$

Comparing the coefficients of z^j, we get the result (29). □

Theorem 5. *Let $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\xi,\eta|\lambda) = \sum_{k=0}^{j}\sum_{s=0}^{k}\binom{j}{k}(x)_s S_2(k,s)\,{}_HD_{j-k}(0,\eta|\lambda). \quad (31)$$

Proof. From (23), we have

$$\sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z+\eta z^2}$$

$$= \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\eta z^2}(e^z-1+1)^{\xi}$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{s=0}^{\infty}(\xi)_s\frac{1}{s!}(e^z-1)^s$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{s=0}^{\infty}(\xi)_s\sum_{k=s}^{\infty}S_2(k,s)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{s=0}^{k}(x)_s S_2(k,s)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{s=0}^{k}\binom{j}{k}(\xi)_s S_2(k,s)\,{}_HD_{j-k}^{(r)}(0,\eta|\lambda)\right)\frac{z^j}{j!}. \quad (32)$$

In view of (32), we get (31). □

Theorem 6. *Let $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\xi+\alpha,y|\lambda) = \sum_{k=0}^{j}\sum_{s=0}^{k}\binom{j}{k}(\xi)_m S_2(k+\alpha,s+\alpha)\,{}_HD_{j-k}(0,\eta|\lambda). \quad (33)$$

Proof. Replacing ξ by $\xi+\alpha$ in (23), we see that

$$\sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi+\alpha,\eta|\lambda)\frac{z^j}{j!} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z+\eta z^2}e^{\alpha z}$$

$$= \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\eta z^2}e^{\alpha z}(e^z-1+1)^{\xi}$$

$$= \sum_{j=0}^{\infty} {}_HD_j(0,\eta|\lambda)\frac{z^j}{j!}e^{\alpha z}\sum_{s=0}^{\infty}(\xi)_m\frac{1}{s!}(e^z-1)^s$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}e^{\alpha z}\sum_{s=0}^{\infty}(\xi)_s\sum_{k=s}^{\infty}S_2(k,s)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{s=0}^{k}(\xi)_sS_2(k+\alpha,s+\alpha)\frac{z^k}{k!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{m=0}^{k}\binom{j}{k}(\xi)_sS_2(k+\alpha,s+\alpha){}_HD_{j-k}^{(r)}(0,\eta|\lambda)\right)\frac{z^j}{j!}. \tag{34}$$

Comparing the coefficients of z, we obtain the result (33). □

Theorem 7. *Let $j\geq 0$. Then*

$${}_HD_j^{(r)}(\xi,\eta|\lambda) = \sum_{l=0}^{j}\binom{j}{l}{}_HD_{j-l}^{(r-k)}(\xi,\eta|\lambda){}_HD_l^{(k)}(0,0|\lambda). \tag{35}$$

Proof. We observe that

$$\sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{\xi z+\eta z^2}$$

$$= \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^{r-k}\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^k e^{\xi z+\eta z^2}$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r-k)}(\xi,\eta|\lambda)\frac{z^j}{j!}\sum_{l=0}^{\infty}{}_HD_l^{(k)}(0,0|\lambda)\frac{z^l}{l!}$$

$$= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}{}_HD_{j-l}^{(r-k)}(\xi,\eta|\lambda){}_HD_l^{(k)}(0,0|\lambda)\right)\frac{z^j}{j!}. \tag{36}$$

Now comparing the coefficients of z, we get the required result. □

Theorem 8. *Let $j\geq 0$. Then*

$${}_HD_j^{(r)}(v,u|\lambda) = \sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi+v,\beta-\eta+u){}_HD_{j-k}^{(r)}(\xi-\alpha,\eta-\beta|\lambda). \tag{37}$$

Proof. By exploiting the generating function (23), we can write

$$\sum_{j=0}^{\infty} {}_HD_n^{(r)}(v,u|\lambda)\frac{z^j}{j!} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{(\xi-\alpha)z+(\eta-\beta)z^2} e^{-(\xi-v-\alpha)z-(\eta-u-\beta)z^2}$$

$$= e^{-(\xi-v-\alpha)z-(\eta-u-\beta)z^2} \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi-\alpha,\eta-\beta|\lambda)\frac{z^j}{j!}$$

$$= \sum_{k=0}^{\infty} H_k(\alpha-\xi+v,\beta-\eta+u)\frac{z^k}{k!} \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi-\alpha,\eta-\beta|\lambda)\frac{z^j}{j!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi+v,\beta-\eta+u){}_HD_{j-k}^{(r)}(\xi-\alpha,\eta-\beta|\lambda)\right)\frac{z^j}{j!}. \quad (38)$$

Comparing the coefficients of z, we arrive at the desired result. □

Remark 1. *Letting $u = v = 0$ in Theorem 8, we get*

Corollary 1. *Let $j \geq 0$. Then*

$$D_j^{(r)}(\lambda) = \sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi,\beta-\eta){}_HD_{j-k}^{(r)}(\xi-\alpha,\eta-\beta|\lambda). \quad (39)$$

Theorem 9. *Let $j \geq 0$. Then*

$$\sum_{q=0}^{j}\sum_{l=0}^{[\frac{j-q}{2}]}\left(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2}\right)^l \frac{{}_HD_{j-2l-q}^{(k)}(\xi,\eta|\lambda)D_q^{(k)}(\lambda)}{m!q!(j-q-2l)!\eta^q\xi^{j-q-2l}}$$

$$= \sum_{l=0}^{j}\frac{D_l^{(k)}(\lambda){}_HD_{j-l}^{(k)}(\eta,\xi|\lambda)}{(j-l)!l!\xi^l\eta^{j-l}}. \quad (40)$$

Proof. On changing z by $\frac{z}{\xi}$ and r by k, we can write (23) as

$$\sum_{j=0}^{\infty} {}_HD_j^{(k)}(\xi,\eta|\lambda)\frac{z^j}{\xi^j j!} = \left(\frac{\log(1+\frac{z}{\xi})}{(1+\log(1+\lambda\frac{z}{\xi}))^{\frac{1}{\lambda}}}\right)^k e^{z+\eta\frac{z^2}{\xi^2}}. \quad (41)$$

Now interchanging ξ by η, we have

$$\sum_{j=0}^{\infty} {}_HD_j^{(k)}(\eta,\xi|\lambda)\frac{z^j}{\eta^j j!} = \left(\frac{\log(1+\frac{z}{y})}{(1+\log(1+\lambda\frac{z}{y}))^{\frac{1}{\lambda}}}\right)^k e^{z+\xi\frac{z^2}{\eta^2}}. \quad (42)$$

Comparison of (41) and (42) yields

$$e^{\xi\frac{z^2}{\eta^2}-\eta\frac{z^2}{\xi^2}}\left(\frac{\log(1+\frac{z}{\xi})}{(1+\log(1+\lambda\frac{z}{\xi}))^{\frac{1}{\lambda}}}\right)^k \sum_{j=0}^{\infty} {}_HD_j^{(k)}(\xi,\eta|\lambda)\frac{z^j}{\xi^j j!}$$

$$= \left(\frac{\log(1+\frac{z}{\xi})}{(1+\log(1+\lambda\frac{z}{\xi}))^{\frac{1}{\lambda}}}\right)^k \sum_{j=0}^{\infty} {}_HD_j^{(k)}(\eta,\xi|\lambda)\frac{z^j}{\eta^j j!}$$

$$= \sum_{l=0}^{\infty} \frac{(\frac{\xi}{\eta^2} - \frac{\eta}{\xi^2})^l}{l!} z^{2l} \sum_{q=0}^{\infty} D_q^{(k)}(\lambda) \frac{z^q}{\eta^q q!} \sum_{j=0}^{\infty} {}_H D_j^{(k)}(\xi, \eta|\lambda) \frac{z^j}{\xi^j j!}$$

$$= \sum_{l=0}^{\infty} D_l^{(k)}(\lambda) \frac{z^l}{\xi^l l!} \sum_{j=0}^{\infty} {}_H D_j^{(k)}(\eta, \xi|\lambda) \frac{z^j}{\eta^j j!}$$

$$\sum_{j=0}^{\infty} \left(\sum_{q=0}^{j} \sum_{l=0}^{[\frac{j-q}{2}]} \left(\frac{\xi}{\eta^2} - \frac{\eta}{\xi^2} \right)^l \frac{{}_H D_{j-2l-q}^{(k)}(\xi, \eta|\lambda) D_q^{(k)}(\lambda)}{l! q! (j-q-2l)! \eta^q \xi^{j-q-2l}} \right) z^j$$

$$= \sum_{j=0}^{\infty} \left(\sum_{l=0}^{j} \frac{D_l^{(k)}(\lambda) {}_H D_{j-l}^{(k)}(\eta, \xi|\lambda)}{(j-l)! l! \xi^l \eta^{j-l}} \right) z^j. \tag{43}$$

By (23) and (43), we obtain the result (40). ☐

3 Implicit formulae involving Appell-type λ-Daehee-Hermite polynomials

We begin by considering the theorems on implicit summation formulae for Appell type λ-Daehee-Hermite polynomials ${}_H D_n^{(r)}(x, y|\lambda)$ and their special cases.

Theorem 10. *Let $j \geq 0$. Then*

$${}_H D_{q+l}^{(r)}(\zeta, \eta|\lambda) \; = \; \sum_{j,p=0}^{q,l} \binom{q}{j} \binom{l}{p} (\zeta - \xi)^{j+p} {}_H D_{q+l-p-j}^{(r)}(\xi, \eta|\lambda). \tag{44}$$

Proof. By changing z by $z + u$ in (23), we get

$$\left(\frac{\log(1+z+u)}{\log(1+\lambda(z+u))^{\frac{1}{\lambda}}} \right)^r e^{\eta(z+u)^2} = e^{-\xi(z+u)} \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\xi, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!}, \tag{45}$$

(Kim et al., 2014; Pathan et al., 2015). Again changing ξ by ζ in (45), we get

$$e^{(\zeta-\xi)(z+u)} \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\xi, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\zeta, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!}. \tag{46}$$

$$\sum_{N=0}^{\infty} \frac{[(\zeta-\xi)(z+u)]^N}{N!} \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\xi, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\zeta, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!}, \tag{47}$$

$$\sum_{N=0}^{\infty} f(N) \frac{(\xi+\eta)^N}{N!} = \sum_{j,m=0}^{\infty} f(j+m) \frac{\xi^j}{j!} \frac{\eta^m}{m!} \tag{48}$$

in the left hand side becomes

$$\sum_{j,p=0}^{\infty} \frac{(\zeta-\xi)^{j+p} z^j u^p}{j! p!} \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\xi, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\zeta, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!} \tag{49}$$

$$\sum_{q,l=0}^{\infty} \sum_{j,p=0}^{q,l} \frac{(\zeta-\xi)^{j+p}}{j! p!} {}_H D_{q+l-j-p}^{(r)}(\xi, \eta|\lambda) \frac{z^q}{(q-j)!} \frac{u^l}{(l-p)!}$$

$$= \sum_{q,l=0}^{\infty} {}_H D_{q+l}^{(r)}(\zeta, \eta|\lambda) \frac{z^q}{q!} \frac{u^l}{l!}. \tag{50}$$

Thus, by (51), we get (45). ☐

Remark 2. *Letting $l = 0$ in Theorem 10, we get*

Corollary 2. *Let $j \geq 0$. Then*

$$ {}_HD_q^{(r)}(\zeta, \eta|\lambda) = \sum_{j=0}^{q} \binom{q}{j} (\zeta - \xi)^j \, {}_HD_{q-j}^{(r)}(\xi, \eta|\lambda). \tag{51} $$

Remark 3. *By changing ζ by $\zeta + \xi$ and taking $\eta = 0$ in Theorem 10, we acquire*

$$ {}_HD_{q+l}^{(r)}(\zeta + \xi|\lambda) = \sum_{j,p=0}^{q,l} \binom{q}{j} \binom{l}{p} \zeta^{j+p} \, {}_HD_{q+l-p-j}^{(r)}(\xi|\lambda). \tag{52} $$

Theorem 11. *Let $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\xi + \zeta, \eta + u|\lambda) = \sum_{s=0}^{j} \binom{j}{s} \, {}_HD_{j-s}^{(r)}(\xi, \eta|\lambda) H_s(\zeta, u). \tag{53} $$

Proof. Rewrite the generating function (23) as

$$ \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r e^{(\xi+\zeta)z + (\eta+u)z^2} = \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi, \eta|\lambda) \frac{z^j}{j!} \sum_{s=0}^{\infty} H_s(\zeta, u) \frac{z^s}{s!} $$

$$ \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi + \zeta, \eta + u|\lambda) \frac{z^j}{j!} $$

$$ = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \binom{j}{s} {}_HD_{j-s}^{(r)}(\xi, \eta|\lambda) H_s(\zeta, u) \right) \frac{z^j}{j!}, \tag{54} $$

which the complete of the proof.

□

Theorem 12. *Let $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\eta, \xi|\lambda) = \sum_{s=0}^{[\frac{j}{2}]} D_{j-2s}^{(r)}(\eta|\lambda) \frac{\xi^s}{(j-2s)!s!}. \tag{55} $$

Proof. By changing ξ by η and η by ξ in (23) to get

$$ \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\eta, \xi|\lambda) \frac{z^j}{j!} = \sum_{j=0}^{\infty} D_j^{(r)}(\eta|\lambda) \frac{z^j}{j!} \sum_{s=0}^{\infty} \frac{\xi^s z^{2s}}{s!}. $$

$$ = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{[\frac{j}{2}]} D_{j-2s}^{(r)}(\eta|\lambda) \frac{\xi^s}{(j-2s)!s!} \right) z^j. $$

The complete of the proof.

□

Theorem 13. *Let $j \geq 0$. Then*

$$ {}_HD_j^{(r)}(\xi, \eta|\lambda) = \sum_{s=0}^{j} \binom{j}{s} D_{j-s}^{(r)}(\xi - \zeta|\lambda) H_s(\zeta, \eta). \tag{56} $$

Proof. By (23), we note that

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{(\xi-\zeta)z}e^{\zeta z+\eta z^2} = \sum_{j=0}^{\infty} D_j^{(r)}(\xi-\zeta|\lambda)\frac{z^j}{j!}\sum_{s=0}^{\infty} H_s(\zeta,\eta)\frac{z^s}{s!}$$

$$\sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!} = \sum_{j=0}^{\infty}\sum_{s=0}^{j} D_{j-s}(x-z|\lambda)H_s(\zeta,\eta)\frac{z^j}{(j-s)!s!}. \tag{57}$$

On equating the coefficients of z, we get the result (57). □

Theorem 14. *Let $j \geq 0$. Then*

$${}_HD_j^{(r)}(\xi+1,\eta|\lambda) = \sum_{s=0}^{j}\binom{j}{s}\xi^s {}_HD_{j-s}^{(r)}(\xi,\eta|\lambda). \tag{58}$$

Proof. In (23), we have

$$\sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi+1,\eta|\lambda)\frac{z^j}{j!} - \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}$$

$$= \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r (e^z-1)e^{\xi z+\eta z^2}$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}\left(\sum_{s=0}^{\infty}\xi^s\frac{z^s}{s!}-1\right)$$

$$= \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}\sum_{s=0}^{\infty}\xi^s\frac{z^s}{s!} - \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty}\sum_{s=0}^{j}\left(\begin{array}{c} j \\ s \end{array}\right)\xi^s {}_HD_{j-s}^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!} - \sum_{j=0}^{\infty} {}_HD_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!},$$

which the complete of the proof.

□

4 A class of two-index real Hermite polynomials and Appell-type λ-Daehee polynomials

This section is a consequence of the definition of the two-index real Hermite-Appell type λ-Daehee polynomials and generalized Gould-Hopper-Appell type λ-Daehee polynomials combined with their properties and special cases.

We define two-index real Hermite-Appell type λ-Daehee polynomials by the generating function beginequation

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{-u^2z^2+(2uz+v)\xi-uvz} = \sum_{j=0}^{\infty} {}_hD_j^{(r)}(\xi,u,v|\lambda)\frac{z^j}{j!}. \tag{59}$$

For $v=0$, (60) reduces to

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^r e^{-u^2z^2+(2uz)\xi} = \sum_{j=0}^{\infty} {}_hD_j^{(r)}(\xi,u,0|\lambda)\frac{z^j}{j!}$$

$$= \sum_{j=0}^{\infty} D_j^{(r)}(\xi|\lambda) \frac{z^j}{j!} \sum_{l=0}^{\infty} H_l(\xi) \frac{u^l z^l}{l!}.$$

Replacing j by $j-l$ and comparing the coefficients of z^j, we get

$${}_hD_j^{(r)}(\xi, u, 0|\lambda) = \sum_{l=0}^{j} \binom{j}{l} u^l D_{j-l}^{(r)}(\xi|\lambda) H_l(\xi),$$

where $H_l(\xi)$ is ordinary Hermite polynomials.

Note that the above result for $u = 1$ is a special case of (23) because when ξ is replaced by 2ξ and $\eta = -1$ then we have

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r e^{2\xi z - z^2} = \sum_{j=0}^{\infty} {}_HD_j^{(r)}(2\xi, -1|\lambda) \frac{z^j}{j!}. \tag{60}$$

In other words

$${}_hD_j^{(r)}(\xi, 1, 0|\lambda) = {}_HD_j^{(r)}(2\xi, -1|\lambda).$$

Theorem 15. *For $j \geq 0$, we have*

$${}_hD_m^{(r)}(\xi, u, v|\lambda) = \sum_{j=0}^{\infty} \sum_{s=0}^{m} D_s^{(r)}(\lambda) h_{m-s,j}(\xi) \frac{u^{m-s} v^j}{(m-s)! j!}. \tag{61}$$

Proof. On replacing u by uz in (8), we have

$$\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} h_{m,j}(\xi) \frac{(uz)^m v^j}{m! j!} = e^{-u^2 z^2 + (2uz+v)\xi - uvz}.$$

Then using (60), we can write

$$\sum_{s=0}^{\infty} {}_hD_s^{(r)}(\xi, u, v|\lambda) \frac{z^s}{s!} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r e^{-u^2 z^2 + (2uz+v)\xi - uvz}$$

$$= \sum_{s=0}^{\infty} D_s^{(r)}(\lambda) \frac{z^s}{s!} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} h_{m,j}(\xi) \frac{(uz)^m v^j}{m! j!}.$$

Now replacing m by $m-s$ and comparing the coefficients of z^s, we get the required result. □

Theorem 16. *Let $j \geq 0$. Then*

$$D_m^{(r)}(\xi\eta|\lambda) = \sum_{j=0}^{\infty} \sum_{s=0}^{m} D_s^{(r)}(\lambda) h_{m-s,j}(\xi) \frac{\eta^{m-s}}{(m-s)! j!}. \tag{62}$$

Proof. We multiply both the sides of (9) by

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r,$$

and replace η by ηz to get

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r e^{\xi\eta z} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r \sum_{m,j=0}^{\infty} (-1)^j h_{m,j}(\xi) \frac{(\eta z)^{m+j}}{m! j!}$$

$$= \sum_{s=0}^{\infty} D_s^{(r)}(\lambda) \frac{z^s}{s!} \sum_{m,j=0}^{\infty} (-1)^j h_{m,j}(\xi) \frac{(\eta z)^{m+j}}{m!j!}.$$

Thus we have

$$\sum_{s=0}^{\infty} D_s^{(r)}(\xi\eta|\lambda) \frac{z^s}{s!} = \sum_{s=0}^{\infty} D_s^{(r)}(\lambda) \frac{z^s}{s!} \sum_{m,j=0}^{\infty} (-1)^j h_{m,j}(\xi) \frac{(\eta z)^{m+j}}{m!j!}.$$

Now replacing m by $m-s$ and comparing the coefficients of z^s, we get the required result. □

Ghanmi & Lamsaf (2019) analyzed a new class of polynomials generalizing different classes of Hermite polynomials such as the real Gould-Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex Hermite polynomials. In the following theorem, we are concerned with a special and unified generalization. More precisely, we deal with the generalized Gould-Hopper polynomials and Appell type λ-Daehee polynomials.

First, we define generalized Gould-Hopper-Appell type λ-Daehee polynomials by the generating function

$$\sum_{j=0}^{\infty} {}_G D_j^{(r)}(w,\gamma,\zeta|u,v|\lambda) \frac{z^j}{j!} = \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r e^{wv+\zeta uz+\gamma u^p v^q}. \tag{63}$$

Note that for $r=0$, (64) reduces to

$$\sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma) \frac{v^m}{m!} e^{\zeta uz} = e^{wv+\zeta uz+\gamma u^p v^q},$$

where $G_m^{(q)}$ is defined by (10).

The next generating function is a consequence of the above one (see Ghanmi & Lamsaf (2019)) and gives the closed expression of $R_\gamma^{p,q}(\zeta,w|u,v)$ in the form

$$R_\gamma^{p,q}(\zeta,w|u,v) = e^{\zeta u+wv+\gamma u^p v^q},$$

where

$$R_\gamma^{p,q}(\zeta,w|u,v) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(\zeta,w|\gamma) \frac{u^j}{j!}\frac{v^m}{m!}. \tag{64}$$

Furthermore, the polynomials $H_{j,m}^{(p,q)}(\zeta,w|\gamma)$ are given by (11) and (12).

Theorem 17. *For every $u,v,w,\zeta,\in\mathbb{C}$ and $j,r\geq 0$, we have*

$${}_G D_j^{(r)}(w,\gamma,\zeta|u,v|\lambda) = \sum_{m=0}^{\infty}\sum_{s=0}^{j} H_{j-s,m}^{(p,q)}(\zeta,w|\gamma) D_s^{(r)}(\lambda) \frac{u^{j-s}v^m}{(j-s)!m!}. \tag{65}$$

Proof. Start with (10), replace γ by $u^p\gamma$ and multiply both the sides by $e^{\zeta uz}$ to get

$$e^{wv+\gamma u^p v^q} e^{\zeta uz} = \sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma) \frac{v^m}{m!} e^{\zeta uz}.$$

Again, we multiply both the sides by

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r,$$

to get

$$e^{wv+\gamma u^p v^q} e^{\zeta uz} \left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}} \right)^r = \sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma) \frac{v^m}{m!} e^{zut} \sum_{s=0}^{\infty} D_s^{(r)}(\lambda) \frac{z^s}{s!}.$$

Thus by using the definitions (64) and (65), we have

$$\sum_{s=0}^{\infty} {}_G D_s^{(r)}(w,\gamma,z|u,v|\lambda) \frac{z^s}{s!}$$

$$= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(\zeta,w|\gamma) \frac{(uz)^j}{j!} \frac{v^m}{m!} \sum_{s=0}^{\infty} D_s^{(r)}(\lambda) \frac{z^s}{s!}. \tag{66}$$

Now replacing j by $j-s$ and comparing the coefficients of z^j, we get the required result. □

As immediate consequence of the above theorem, we have

Corollary 3. *For every $u, v, w, \zeta, \in \mathbb{C}$ and $j \geq 0$, we have*

$${}_G D_j^{(0)}(w,\gamma,\zeta|u,v|\lambda) = R_\gamma^{p,q}(\zeta,w|u,v) = e^{\zeta u + wv + \gamma u^p v^q}.$$

5 Symmetry identities for Appell-type λ-Daehee-Hermite polynomials

In this section, we give general symmetry identities for Appell-type λ-Daehee-Hermite polynomials ${}_H D_j^{(r)}(\xi,\eta|\lambda)$ by applying the generating functions (4) and (23).

Theorem 18. *For $j \geq 0$. Then*

$$\sum_{s=0}^{j} \binom{j}{s} a^{j-s} b^s {}_H D_{j-s}^{(r)}\left(b\xi, b^2\eta|\lambda\right) {}_H D_s^{(r)}\left(a\xi, a^2\eta|\lambda\right)$$

$$= \sum_{s=0}^{j} \binom{n}{s} b^{j-s} a^s {}_H D_{j-s}^{(r)}\left(a\xi, a^2\eta|\lambda\right) {}_H D_s^{(r)}\left(b\xi, b^2\eta|\lambda\right). \tag{67}$$

Proof. Let

$$A(z) = \left(\frac{\log(1+az)\log(1+bz)}{(\log(1+\lambda z)^{\frac{a}{\lambda}})(\log(1+\lambda z)^{\frac{b}{\lambda}})} \right)^r e^{ab\xi z + a^2 b^2 \eta z^2}.$$

Then the expression for $A(z)$ is symmetric in a and b and we can expand $A(z)$ into series in two ways to obtain

$$A(z) = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \binom{j}{s} a^{j-s} b^s {}_H D_{j-s}^{(r)}\left(b\xi, b^2\eta|\lambda\right) {}_H D_s^{(r)}\left(a\xi, a^2\eta|\lambda\right) \right) \frac{z^j}{j!}.$$

On the similar lines we can show that

$$A(z) = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \binom{j}{s} b^{j-s} a^s {}_H D_{j-s}^{(r)}\left(a\xi, a^2\eta|\lambda\right) {}_H D_s^{(r)}\left(b\xi, b^2\eta|\lambda\right) \right) \frac{z^j}{j!},$$

which implies the desired result. □

Remark 4. *Letting $b = 1$ in Theorem 18, we get*

Corollary 4. *Let $j \geq 0$. Then*

$$\sum_{s=0}^{j} \binom{j}{s} a^{j-s} {}_H D_{j-s}^{(r)} (\xi, \eta|\lambda) \, {}_H D_s^{(r)} \left(a\xi, a^2\eta|\lambda\right)$$

$$= \sum_{s=0}^{j} \binom{j}{s} a^s \, {}_H D_{j-s}^{(r)} \left(a\xi, a^2\eta|\lambda\right) {}_H D_s^{(r)} (\xi, \eta|\lambda) . \tag{68}$$

Theorem 19. *Let $j \geq 0$. Then*

$$\sum_{s=0}^{j} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} \binom{j}{s} a^{j-s} b^s \, {}_H D_{j-s}^{(r)} \left(b\eta + \frac{b}{a} i + l, b^2\zeta|\lambda\right) D_s^{(r)}(a\eta|\lambda)$$

$$= \sum_{s=0}^{j} \sum_{l=0}^{a-1} \sum_{i=0}^{b-1} \binom{j}{s} b^{j-s} a^s \, {}_H D_{j-s}^{(r)} \left(a\eta + \frac{a}{b} i + l, a^2\zeta|\lambda\right) D_s^{(r)}(b\eta|\lambda). \tag{69}$$

Proof. Let

$$B(z) = \left(\frac{\log(1+az)\log(1+bz)}{(\log(1+\lambda z)^{\frac{a}{\lambda}})(\log(1+\lambda z)^{\frac{b}{\lambda}})} \right)^r \frac{(e^{abz}-1)^2}{(e^{az}-1)(e^{bz}-1)} e^{ab(\xi+\eta)z + a^2b^2\zeta z^2}$$

$$= \left(\frac{\log(1+az)}{\log(1+\lambda z)^{\frac{a}{\lambda}}} \right)^r e^{ab\xi z + a^2b^2\zeta z^2} \sum_{i=0}^{a-1} e^{bzi} \left(\frac{\log(1+bz)}{\log(1+\lambda z)^{\frac{b}{\lambda}}} \right)^r e^{ab\eta z} \sum_{l=0}^{b-1} e^{azj} \tag{70}$$

$$= \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} \binom{j}{s} a^{j-s} b^s \, {}_H D_{j-s}^{(r)} \left(b\xi + \frac{b}{a} i + l, b^2\zeta|\lambda\right) D_s^{(r)}(a\eta|\lambda) \right) \frac{z^j}{j!}.$$

On the other hand, we have

$$B(z) = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \sum_{l=0}^{a-1} \sum_{i=0}^{b-1} \binom{j}{s} b^{j-s} a^s \, {}_H D_{j-s}^{(r)} \left(a\xi + \frac{a}{b} i + l, a^2\zeta|\lambda\right) D_s^{(r)}(b\eta|\lambda) \right) \frac{z^j}{j!},$$

which provide the desired result. □

Theorem 20. *Let $j \geq 0$. Then*

$$\sum_{s=0}^{j} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} \binom{j}{s} a^{j-s} b^s \, {}_H D_{j-s}^{(r)} \left(b\eta + \frac{b}{a} i, b^2\zeta|\lambda\right) D_s^{(r)} \left(a\eta + \frac{a}{b} l|\lambda\right)$$

$$= \sum_{s=0}^{j} \sum_{l=0}^{a-1} \sum_{i=0}^{b-1} \binom{j}{s} b^{j-s} a^s \, {}_H D_{j-s}^{(r)} \left(a\eta + \frac{a}{b} i, a^2\zeta|\lambda\right) D_s^{(r)} \left(b\eta + \frac{b}{a} l|\lambda\right). \tag{71}$$

Proof. By (70), we note that

$$B(z) = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \sum_{i=0}^{a-1} \sum_{l=0}^{b-1} \binom{j}{s} a^{j-s} b^s \, {}_H D_{j-s} \left(b\eta + \frac{b}{a} i, b^2\zeta|\lambda\right) D_m \left(a\eta + \frac{a}{b} l|\lambda\right) \right) \frac{z^j}{j!}.$$

On the other hand Equation (70) can be shown equal to

$$B(z) = \sum_{j=0}^{\infty} \left(\sum_{s=0}^{j} \sum_{l=0}^{a-1} \sum_{i=0}^{b-1} \binom{j}{s} b^{j-s} a^s \, {}_H D_{j-s}^{(r)} \left(a\eta + \frac{a}{b} i, a^2\zeta|\lambda\right) D_s^{(r)} \left(b\eta + \frac{b}{a} l|\lambda\right) \right) \frac{z^j}{j!},$$

which implies the desired result. □

Now, we prove the following symmetric identity involving sum of integer powers $S_k(n)$ given by equation (21), Appell-type λ-Daehee-Hermite polynomials ${}_HD_j^{(r)}(\xi,\eta|\lambda)$.

Theorem 21. *Let $j \geq 0$. Then*

$$\sum_{k=0}^{j}\binom{j}{k}a^{j-k}b^{k}{}_HD_{j-k}^{(r)}\left(b\xi,b^2\zeta|\lambda\right)\sum_{i=0}^{k}\binom{k}{i}S_i(b-1)D_{k-i}^{(r)}(a\eta|\lambda)$$

$$\sum_{k=0}^{j}\binom{j}{k}b^{j-k}a^{k}{}_HD_{j-k}^{(r)}\left(a\xi,a^2\zeta|\lambda\right)\sum_{i=0}^{k}\binom{k}{i}S_i(a-1)D_{k-i}^{(r)}(b\eta|\lambda). \tag{72}$$

Proof. Let

$$C(z)=\left(\frac{\log(1+az)\log(1+bz)}{(\log(1+\lambda z)^{\frac{a}{\lambda}})(\log(1+\lambda z)^{\frac{b}{\lambda}})}\right)^{r}\frac{(e^{abz}-1)}{(e^{az}-1)(e^{bz}-1)}e^{ab(\xi+\eta)z+a^2b^2\zeta z^2}$$

$$=\sum_{j=0}^{\infty}{}_HD_j^{(r)}\left(b\xi,b^2\zeta|\lambda\right)\frac{(az)^j}{j!}\sum_{i=0}^{\infty}S_i(b-1)\sum_{k=0}^{\infty}D_k^{(r)}(a\eta|\lambda)\frac{(bz)^k}{k!}$$

$$=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}a^{j-k}b^{k}{}_HD_{j-k}^{(r)}\left(b\xi,b^2\zeta|\lambda\right)\sum_{i=0}^{k}\binom{k}{i}S_i(b-1)D_{k-i}^{(r)}(a\eta|\lambda)\right)\frac{z^j}{j!}.$$

On the other hand

$$C(z)=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}b^{j-k}a^{k}{}_HD_{j-k}^{(r)}\left(a\xi,a^2\zeta|\lambda\right)\sum_{i=0}^{k}\binom{k}{i}S_i(a-1)D_{k-i}^{(r)}(b\eta|\lambda)\right)\frac{z^j}{j!}.$$

The complete proof of this theorem. ◻

6 Concluding remarks

So far, we have given the definition of the Appell type λ-Daehee-Hermite polynomials, their explicit forms, miscellaneous properties and certain symmetry identities. Before closing the paper, let us add some comments aimed at better framing the obtained results. Using (6), we have extended (22) in the form of a generating function (23). With the help of Dattoli's idea on the truncated polynomials (see Dattoli et al. (2003)), our polynomials named as λ-Daehee-truncated Hermite polynomials can be extended to a newly constructed form as follows:

$$\left(\frac{\log(1+z)}{\log(1+\lambda z)^{\frac{1}{\lambda}}}\right)^{r}e_m(\xi z+\eta z^2)=\sum_{j=0}^{\infty}{}_HD_j^{(r)}(\xi,\eta|\lambda,m)\frac{z^j}{j!}, \tag{73}$$

where $e_m(\xi)$ is the truncated exponential function defined by Dattoli et al. (2003)

$$e_m(\xi)=\sum_{j=0}^{m}\frac{\xi^j}{j!},\qquad \frac{e^{\xi z}}{1-z}=\sum_{j=0}^{\infty}z^je_j(\xi),|z|<1. \tag{74}$$

The reason of the interest for this family of polynomials stems from the fact that they currently appear in the theory of the so-called flattened beams, which plays a role of paramount importance in optics and in particular in the case of super-Gaussian optical resonators (see Gori (1994)). Truncated polynomials have also been studied in the evaluation of integrals involving products of special functions.

Hence, it may be interesting to study such topics in the future.

We may also discuss the properties of the truncated exponential polynomials and develop the theory of new form of Hermite polynomials, namely,

$$\sum_{j=0}^{\infty} H_j(\xi,\eta|m)\frac{z^j}{j!} = e_m(\xi z+\eta z^2), \tag{75}$$

which can be constructed using the truncated exponential as a generating function. We can derive their explicit forms and comment on their usefulness in applications, with particular reference to the theory of flattened beams, used in optics.

References

Andrews, L.C. (1985). S*pecial functions for engineers and applied mathematicians.* Macmillan. New York.

Bell, E.T. (1934). Exponential polynomials. *Annals of Mathematics*, 258-277.

Dattoli, G., Chiccoli, C., Lorenzutta, S., Maino, G., & Torre, A. (1994). Theory of generalized Hermite polynomials. *Computers & Mathematics with Applications, 28*(4), 71-83.

Dattoli, G., Cesarano, C., & Sacchetti, D. (2003). A note on truncated polynomials. *Applied Mathematics and Computation, 134*(2-3), 595-605.

Deeba, E.Y., Rodriguez, D.M. (1991). Stirling's series and Bernoulli numbers. *The American Mathematical Monthly, 98*(5), 423-426.

Gori, F. (1994). Flattened Gaussian beams. *Optics Communications, 107*(5-6), 335-341.

Ghanmi, A., Lamsaf, K. (2019). Two-dimensional (p, q)-heat polynomials of Gould–Hopper type. arXiv preprint arXiv:1907.10744.

Gould, H.W., Hopper, A.T. (1962). Operational formulas connected with two generalizations of Hermite polynomials. *Duke Mathematical Journal, 29*(1), 51-63.

Jedda, N.A, Ghanmi, A. (2014). On a class of two-index real Hermite polynomials. *Palestine Journal of Mathematics, 3*(2) 185-190.

Khan, W. A., Muhiuddin, G., Muhyi, A., & Al-Kadi, D. (2021). Analytical properties of type 2 degenerate poly-Bernoulli polynomials associated with their applications. *Advances in Difference Equations, 2021*(1), 1-18.

Khan, W. A., Muhyi, A., Ali, R., Alzobydi, K.A.H., Singh, M., & Agarwal, P. (2021). A new family of degenerate poly-Bernoulli polynomials of the second kind with its certain related properties. *AIMS Mathematics, 6*(11), 12680-12697.

Khan, W.A., Acikgoz, M., & Duran, U. (2020). Note on the type 2 degenerate multi-poly-Euler polynomials. *Symmetry, 12*(10), 1691.

Khan, W.A. (2022). A new class of higher-order hypergeometric Bernoulli polynomials associated with Hermite polynomials. *Boletim da Sociedade Paranaense de Mathematica, 40*, 1-14.

Khan, W.A., Kamarujjama, M., & Daud. (2022). Construction of partially degenerate Bell-Bernoulli polynomials of the first kind. *Analysis (Germany), 43*(3), 1-14.

Kim, D.S. (2013). A Note on the λ-Daehee polynomials. *Int. J. Math. Anal., 7*(62) 3069-3080.

Kim, T., Lee, S., Mansour, T. & Seo, J.J. (2014). A note on q-Daehee polynomials and numbers. *Adv. Stud. Contemp. Math. (Kyungshang)*, *24*(2), 155-160.

Kwon, J.K., Rim, S.H., Park, J.W. (2015). A note on the Appell-type Daehee polynomials. *Global J. Pure and Applied Math., 11*(5), 2745-2753.

Pathan, M.A., Khan, W.A. (2015). Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials. *Mediterr. J. Math., 12*, 679-695.

Pathan, M. A and Khan, W.A. (2016). A new class of generalized polynomials associated with Hermite and Euler polynomials. *Mediterr. J. Math.*, 13, 913-928.

Pathan, M.A., Khan, W.A. (2021). A new class of generalized polynomials associated with Hermite and poly-Bernoulli polynomials. *Miskolc Math. J.*, *22*(1), 317-330.

Pathan, M.A., Khan, W.A. (2022). On *lambda*-Changhee-Hermite polynomials. *Analysis (Germany)*, *42*(2), 57-69.

Pathan, M.A., Khan, W.A. (2022). On a class of generalized Humbert-Hermite polynomials via generalized Fibonacci polynomials. *Turkish Journal of Mathematics*, *46*, 929-945.

San Kim, D., Kim, T., Lee, S.H., & Seo, J.J. (2013). A note on the twisted λ-Daehee polynomials. *Applied Mathematical Sciences, 7*(141), 7005-7014.